Fixed, Random, and Mixed Effects

John Poe, University of Kentucky

Overview

- What Are We Talking About?
- The Tower of Babel
- Demons and Monsters
- How To Beat the Demons and Monsters
- A Digression on Not Being a Sheep

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Basic models assume data are conditionally independent

Most social science data are clustered

People

People in the same place

People under the same government

People interacting with one another

People exposed to the same things

People with similar lived experiences

Is this a problem for us?

Maybe

Probably

Okay, yes definitely

Why?

There are many ways to deal with this sort of problem

Spatial models

Network models

Panel models

Longitudinal models

Hierarchical models

Many of these things can be collapsed into multilevel or mixed effects models

Advice on what models apply when and how is often confusing and contradictory

Why?

There are three fundamental types of statistical analysis

Modeling a DGP

Causal inference about a mechanism

Prediction

Clustering issues affect each of these goals in different ways

Modeling a DGP properly usually requires that you take into account group-conditionality

Causal inference either controls for group conditionality or redefines the mechanism in question to be conditional

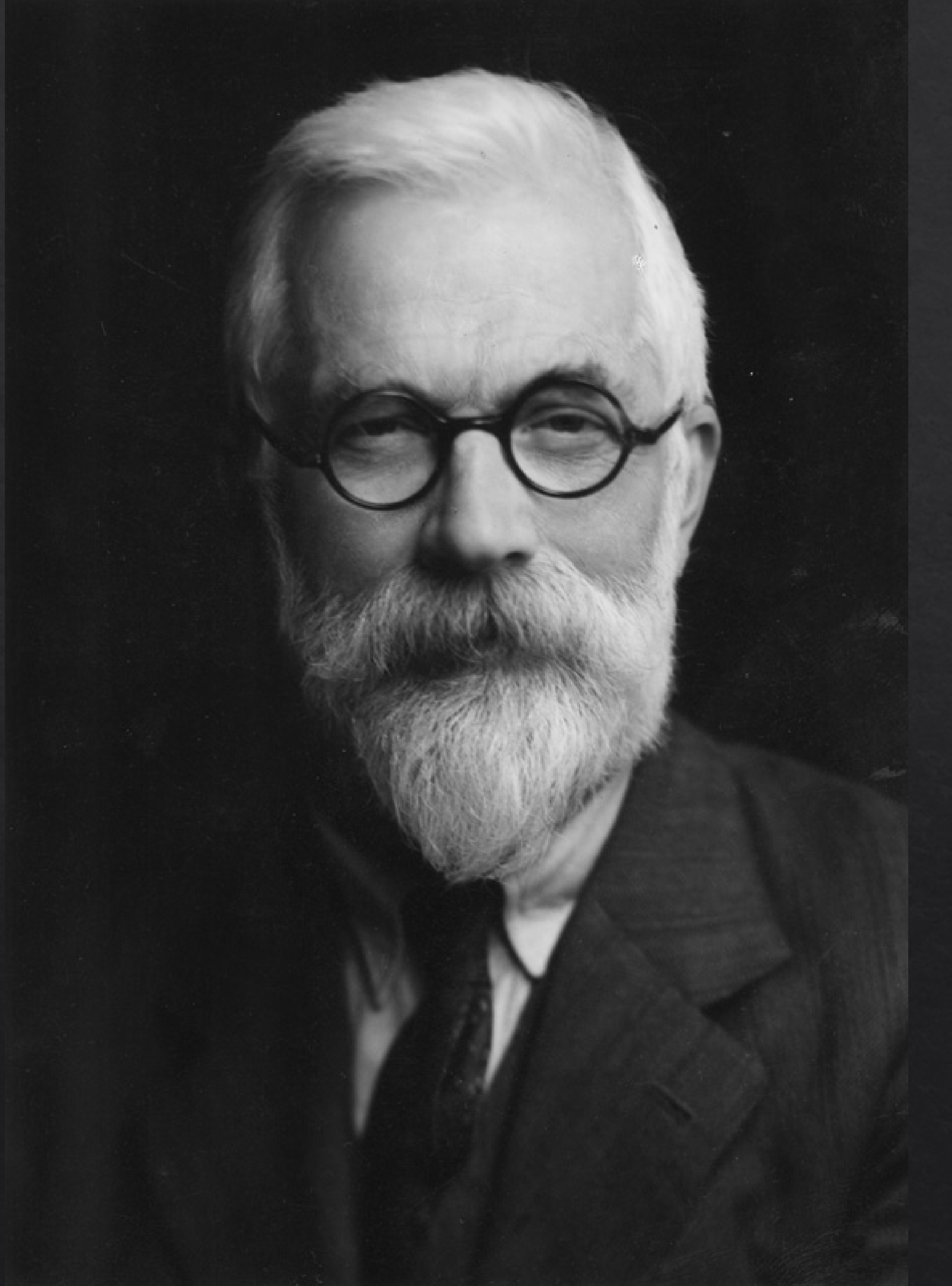
Predictive accuracy almost always benefits from accounting for group conditionality

These goals interrelate

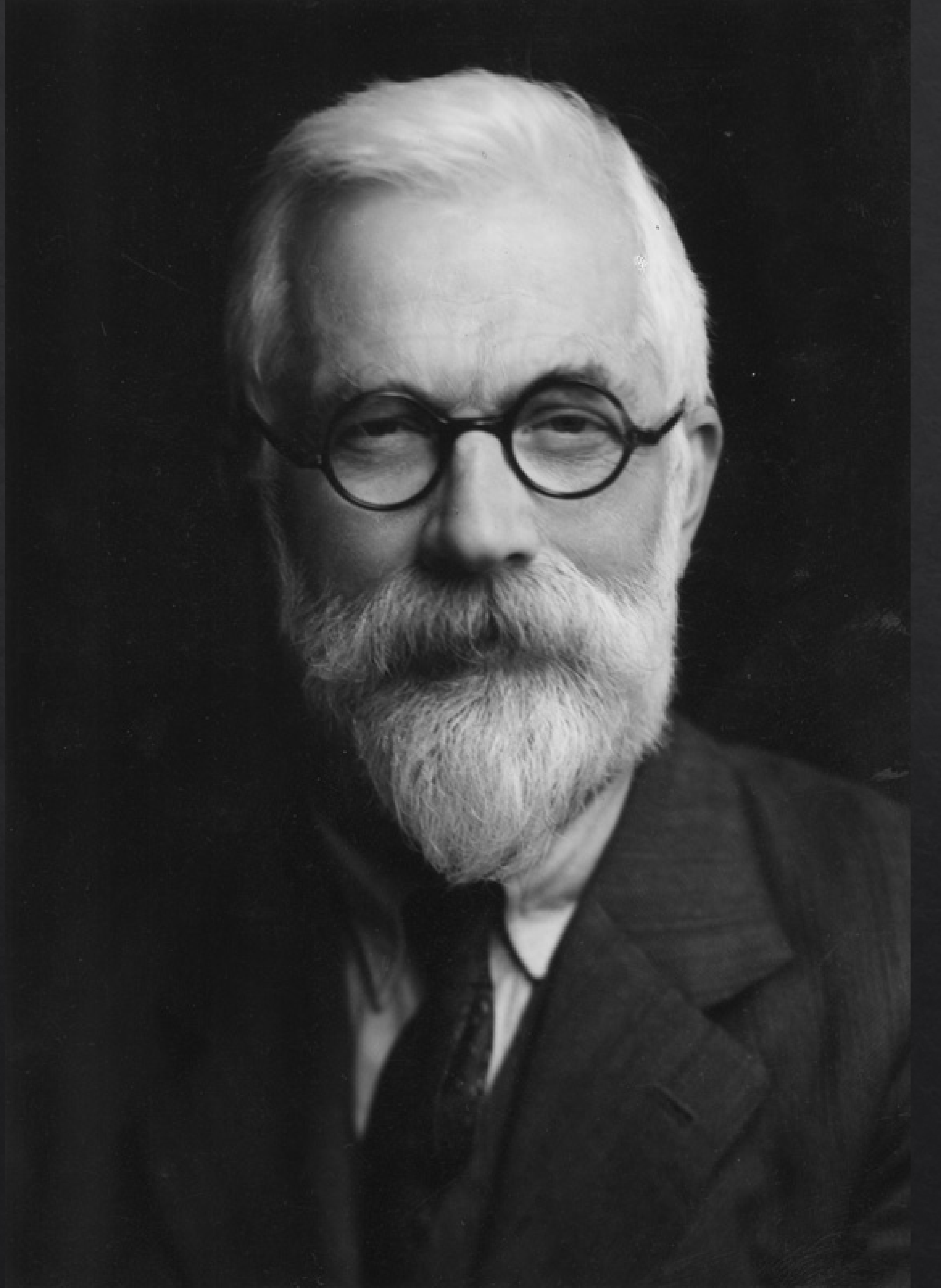
Once upon a time there was a common language to statistics

The field was new and words meant what they meant

R.A. Fisher taught people ANOVA



People liked ANOVA



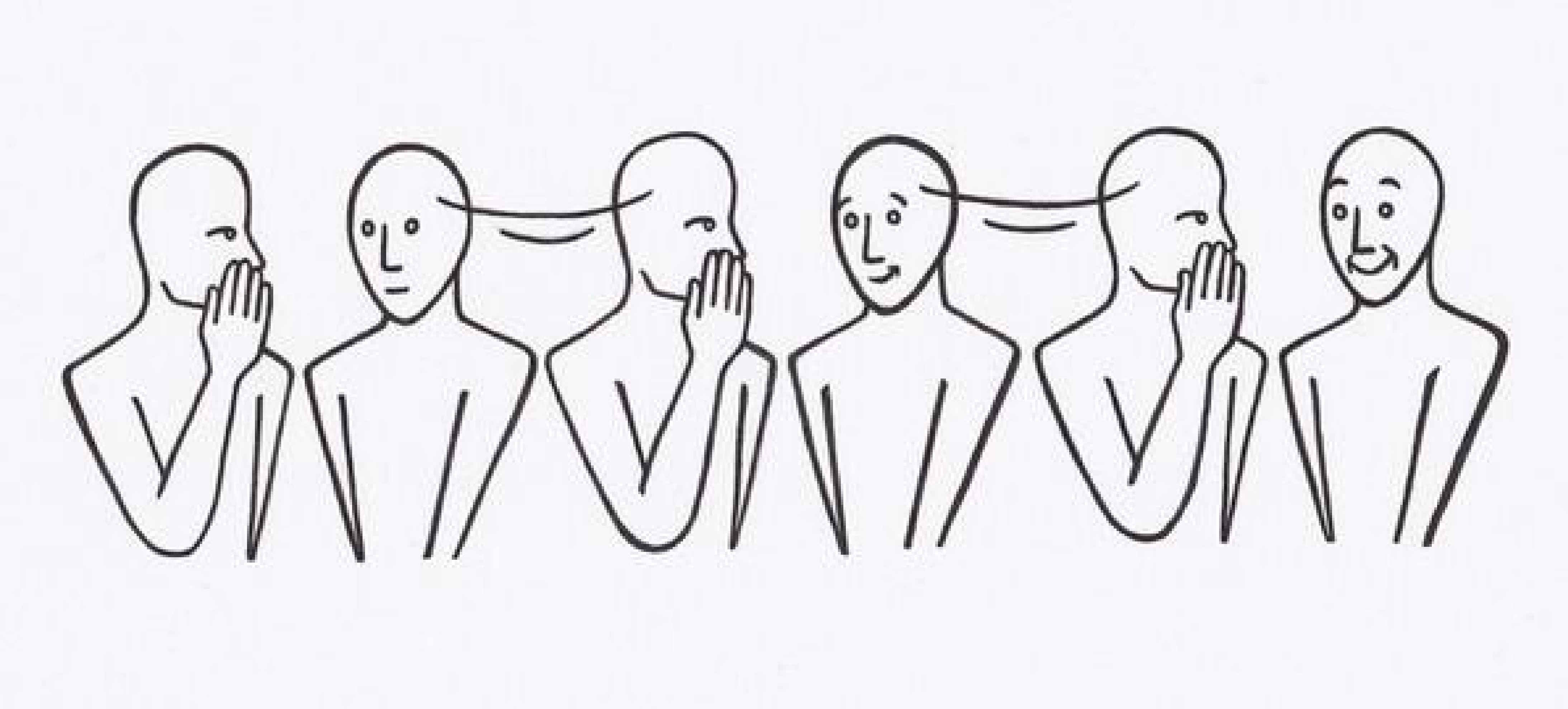
Henry Scheffé wrote a whole book

A fixed effect was a specific group comparison

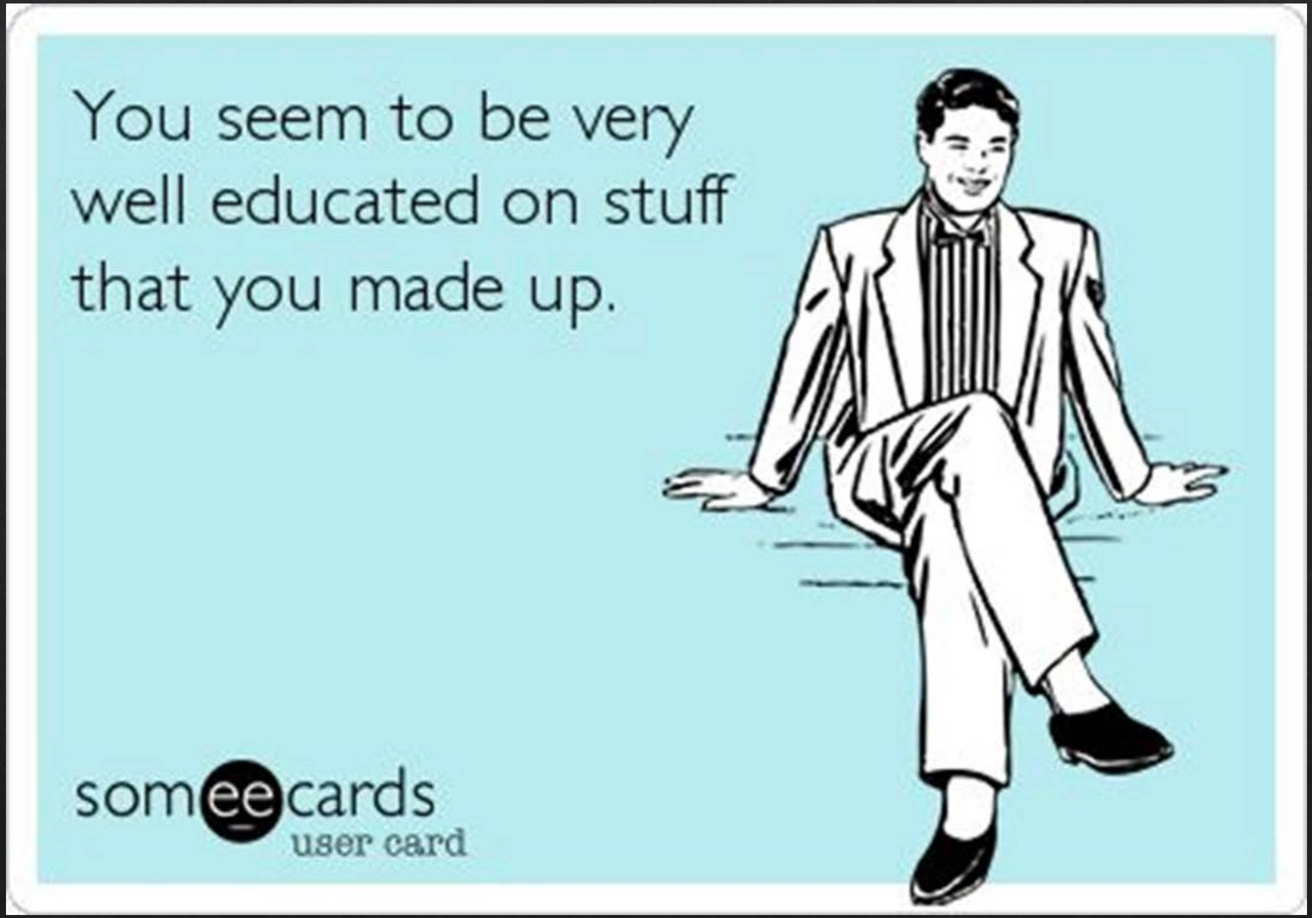
A random effect was a standard error correction

Then the tower started to collapse

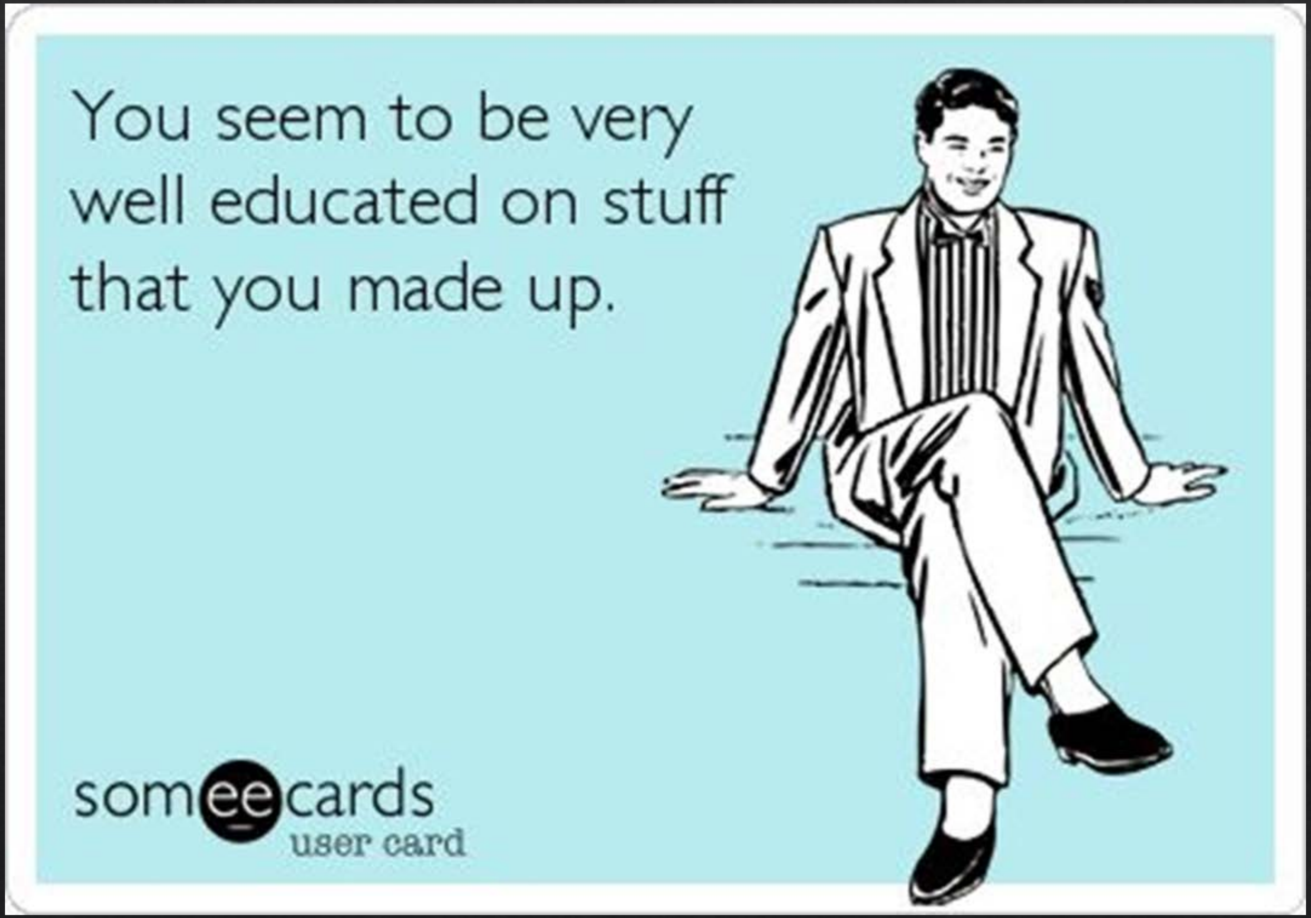
Other fields started to borrow fixed and random effects



Terms started mutating



The definitions tend to proliferate



Now “fixed” and “random” mean very different things to different people

Fixed effects: a type of *model* using only within group variability to estimate model parameters

Fixed effects: *variables* that do not vary randomly across groups

Fixed effects: *coefficients* on within group varying variables

Fixed effects: *dummy variables* used to remove between group variability

Random effects: a latent *variable* made up of the expected values of Y based on group membership

Random effects: any *variable* that is allowed to vary across groups within a model

Random effects: *the variance* around the model intercept when that intercept is allowed to vary across groups

Random effects: *the variance* around any variable that is that is allowed to vary across groups within a model

Random effects: a *class of models* where you allow some parameters to vary across groups

Random effects: *a type of model* that causes endogeneity and is basically evil

Mixed effects: *a type of model* that has both random effects and fixed effects

So if someone says fixed & random effects they mean:

- a variable
- a coefficient
- the variance on a coefficient
- multiple variables
- multiple coefficients
- multiple variances around multiple coefficients
- a specific model
- or an entire class of models

That’s not even mentioning Bayesian random effects

Pretty straightforward, right?

For our purposes a fixed effect will be things like α and β

For our purposes a random effect will be things like μ

Fixed Effects *or* Random Effects models are different

Why should **you** care *at all* about this?

There are distinct but interrelated problems that clustered data can cause in an analysis

Omitted variable bias messing with the standard errors

Cluster confounding

Spatial diffusion across groups

Network diffusion across groups

Temporal diffusion across groups

Selection effects

Missing levels

All of which are special cases of omitted variable bias

All of which are special cases of endogeneity

The effect of X on Y is inconsistent depending on group membership

What does this mean?!?

Your standard errors are probably wrong

Your coefficients are probably wrong

You probably don’t have the right variables in your model

You probably aren’t even testing your hypotheses

If you try to predict fitted values

If you try to predict probabilities

If you try to predict propensity scores

They are likely biased

Questions?

How do we kill the beast?

Different solutions exist for different problems

Standard errors are easy to fix

Hubert-White Cluster Robust Standard Errors

Cluster Bootstrapped/Jackknifed Standard Errors

Including a random effect in the model

Endogeneity is much more complicated

Hack out the affected parts

Allow the effect of X on Y to vary and get on with your life

Model the sources of endogeneity

You have six basic options of equations with various bells and whistles on them in the literature(s)

$$Y_i = \alpha + \beta(X_i) + \varepsilon_i$$

A Classical Linear Regression Model

The fixed effects within the model

$$Y_i = \alpha + \beta(X_i) + \varepsilon_i$$

A Classical Linear Regression Model

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon_i$$

Economists:	A Fixed Effects Model
Statisticians:	Very Inefficient
Psychologists:	o_O

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$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu_j + \varepsilon_i$$

Economists:	A Random Effects Model
Statisticians:	A Random Intercept Model
Psychologists:	A Random Intercept Model

The fixed effects within the model

The random effects within the model

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu_j + \varepsilon_i$$

Economists:	A Random Effects Model
Statisticians:	A Random Intercept Model
Psychologists:	A Random Intercept Model

$$\boldsymbol{Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu_j + \varepsilon_i}$$

Economists:	A Mundlak Device?
Economists:	Correlated Random Effects
Statisticians:	Group Mean Centering
Psychologists:	Group Mean Centering

The fixed effects within the model

The random effects within the model

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu_j + \varepsilon_i$$

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$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu_j + \mu_j(X_{ij} - \bar{X}_j) + \varepsilon_i$$

Everyone: A Random Coefficients Model
Everyone: A Random Slopes Model
Everyone: A Varying Coefficients Model

The fixed effects within the model

The random effects within the model

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu_j + \mu_j(X_{ij} - \bar{X}_j) + \varepsilon_i$$

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$$\boldsymbol{Y_{ij}} = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{\beta}\boldsymbol{\bar{X}_j} + \boldsymbol{\mu_j} + \boldsymbol{\mu_j}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{N_j} + \boldsymbol{N_j}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{S_j} + \boldsymbol{S_j}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{T_j} + \boldsymbol{T_j}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{Z_j} + \boldsymbol{Z_j}(\boldsymbol{X_{ij}} - \boldsymbol{\bar{X}_j}) + \boldsymbol{\varepsilon_i}$$

STATA: OMGOMGOMGOMGOMGOMG
LME4: HAHAHAHAHAHAHAHAHAHA
BRMS: Let’s do this

The fixed effects within the model

The random effects within the model

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu_j + \mu_j(X_{ij} - \bar{X}_j) + N_j + N_j(X_{ij} - \bar{X}_j) + S_j + S_j(X_{ij} - \bar{X}_j) + T_j + T_j(X_{ij} - \bar{X}_j) + Z_j + Z_j(X_{ij} - \bar{X}_j) + \varepsilon_i$$

STATA: OMGOMGOMGOMGOMG

LME4: HAHAHAHAHAHAHAHAHA

BRMS: Let's do this

Now Bayes!

$$\boldsymbol{Y_i = \alpha + \beta(X_i) + \varepsilon}$$

A Classical Linear Regression Model

The fixed effects within the model

$$Y_i = \alpha + \beta(X_i) + \varepsilon$$

A Classical Linear Regression Model

The random effects within the model

$$Y_i = \alpha + \beta(X_i) + \varepsilon$$

A Bayesian Linear Regression Model

The random effects within the model

$$Y_i = \Theta + \Theta(X_i) + \varepsilon$$

A Bayesian Linear Regression Model

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Economists:	A Random Effects Model
Statisticians:	A Random Intercept Model
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The fixed effects within the model

The random effects within the model

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The random effects within the model

The random effects within the model

$$\boldsymbol{Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon}$$

Economists:	An abomination
Statisticians:	A Bayesian RI Model
Psychologists:	Can I do an ANOVA instead?

The random effects within the model

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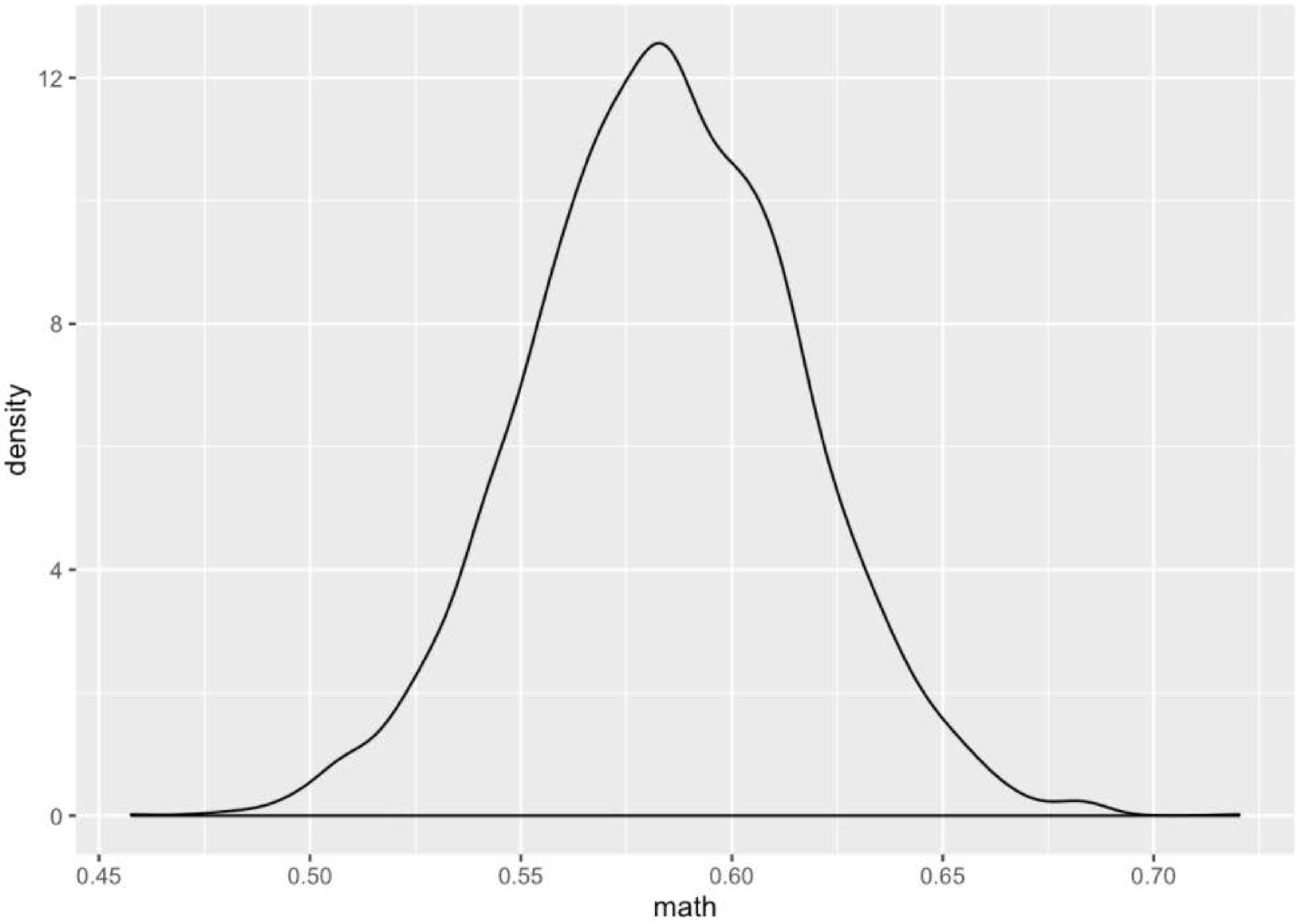

12
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4
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density
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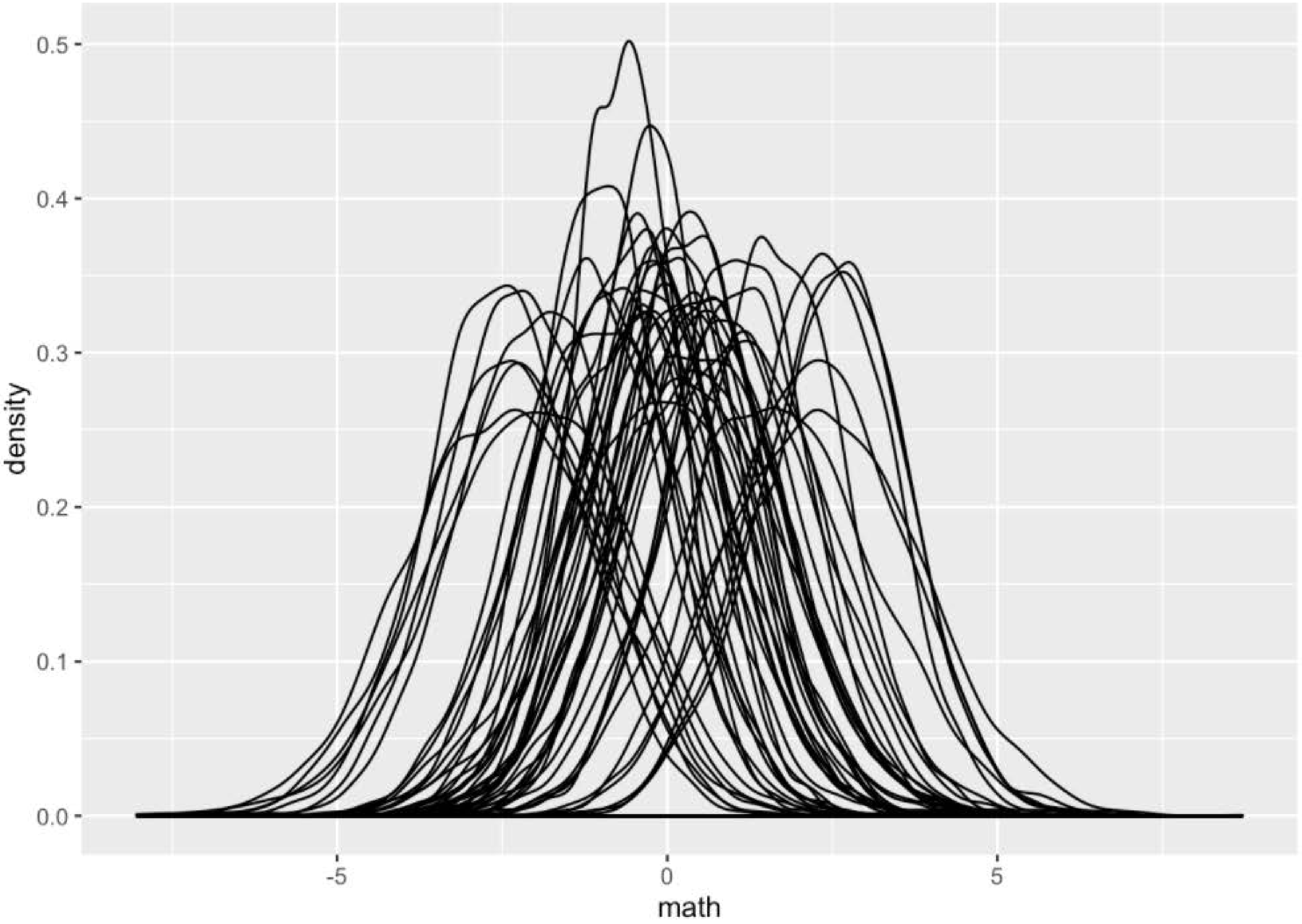
0.50
0.55
0.60
0.65
0.70
math



0.5
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density
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math

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

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Economists:	Correlated Random Effects
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The random effects within the model

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Everyone: A Varying Coefficients Model

Your choice is based on your *hypotheses* and your *data*

Hypothesis-based Reasons to use Fixed Effects

Use Fixed Effects

Hack out all between group variance and throw it away

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

You only care about within group variability and not between group variability

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

Maybe don’t throw it away and compare specific groups

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

Your hypotheses are about within person change over time or average within group effects controlling for average group differences

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

You do **not** want to make predictions

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Data-based Reasons to use Fixed Effects

Use Fixed Effects

You don't have many groups

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

You can't figure out how to specify the right kind of model with random effects

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Use Fixed Effects

A random effects model isn't computationally stable

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Sociological Reasons to use Fixed Effects

Use Fixed Effects

An economist or someone trained by one will review your paper and you don't want it rejected

$$Y_{ij} = \alpha + \beta(X_i) + \beta(J_1) + \beta(J_2) + \cdots \beta(J_{g-1}) + \varepsilon$$

Hypothesis-based Reasons to use Random Effects

Use Random Effects

If you have hypotheses about group level variables that are time or group invariant

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Data-based Reasons to use Random Effects

Use Random Effects

You have no correlation between independent variables and the random effect*

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Use Random Effects

You traveled back in time to the 1970s and need a more efficient estimator than fixed effects

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Use Random Effects

You are running a nonlinear maximum likelihood model and want to improve model specification

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Sociological Reasons to use Random Effects

Use Random Effects

You enjoy being yelled at by economists

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Use Random Effects

You don't like other standard error fixes

$$Y_{ij} = \alpha + \beta(X_{ij}) + \mu + \varepsilon$$

Hypothesis-based Reasons to use Group-Mean Centering

Use Group-Mean Centering

You care about understanding contextual effects

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You care about understanding group-level variables

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You care about understanding within-group effects

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You care about understanding cross-level effects

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You want to make predictions

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Data-based Reasons to use Group-Mean Centering

Use Group-Mean Centering

You want to do a Fixed Effects model but you have a (very) nonlinear outcome and small within group samples

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You don't have much within group variability

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You are having convergence issues in MLE or MCMC

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Sociological Reasons to use Group-Mean Centering

Use Group-Mean Centering

You are going to be reviewed by a psychologist and want to get published

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Use Group-Mean Centering

You are going to be reviewed by an economist and can’t use dummies or differences.

Call it correlated random effects

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \varepsilon$$

Hypothesis-based Reasons to use Random Coefficients

Use Random Coefficients

You care about understanding how the effect of an independent variable varies across groups

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \mu(X_{ij} - \bar{X}_j) + \varepsilon$$

Use Random Coefficients

You want to make predictions

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \mu(X_{ij} - \bar{X}_j) + \varepsilon$$

Use Random Coefficients

You want to understand context effects

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \mu(X_{ij} - \bar{X}_j) + \varepsilon$$

Data-based Reasons to use Random Coefficients

Use Random Coefficients

The Mundlak device still isn't getting you unbiased within group coefficients

$$\boldsymbol{Y}_{ij} = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_j) + \boldsymbol{\beta}\bar{\boldsymbol{X}}_j + \boldsymbol{\mu} + \boldsymbol{\mu}(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_j) + \boldsymbol{\varepsilon}$$

Use Random Coefficients

You want to know how people are different across different contexts

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \mu(X_{ij} - \bar{X}_j) + \varepsilon$$

Sociological Reasons to use Random Coefficients

Use Random Coefficients

There aren't really any but there should be

$$Y_{ij} = \alpha + \beta(X_{ij} - \bar{X}_j) + \beta\bar{X}_j + \mu + \mu(X_{ij} - \bar{X}_j) + \varepsilon$$

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Panel/Longitudinal

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Random Coefficients

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- Wooldridge, Jeffrey M. 2010. *Econometric Analysis of Cross Section and Panel Data*. Second ed. Chapter 4.3.3 and 6.4